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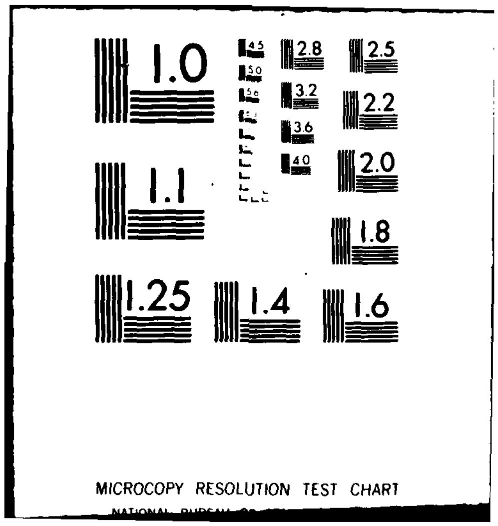
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Statistical Inference for a New Model in Reliability

Ronald E. Glaser

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Division of Statistics
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A new family of life models proposed by Glaser (1980) has densities of the form $C(\theta_1,\theta_2,\theta_3)\exp\{-\theta_1 t - \theta_2 t^2 + \theta_3 \log t\}$. The family includes all gamma distributions, all exponential distributions, all normal distributions left truncated at zero, and a variety of distributions having bathtub shaped failure rate functions. In this paper statistical inference for the family is considered with an emphasis on determining whether the sampled model fits the bathtub category.

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ABSTRACT

A new family of life models proposed by Glaser (1980) has densities of the form $C(\theta_1,\theta_2,\theta_3)\exp\{-\theta_1 t - \theta_2 t^2 + \theta_3 \log t\}$. The family includes all gamma distributions, all exponential distributions, all normal distributions left truncated at zero, and a variety of distributions having bathtub shaped failure rate functions. In this paper statistical inference for the family is considered with an emphasis on determining whether the sampled model fits the bathtub category.

1. INTRODUCTION

A family $\mathcal{F}$ of life distributions has been proposed by Glaser (1980). It is characterized by the collection of density functions of the form

$$f(t) = C(\theta_1,\theta_2,\theta_3)\exp\{-\theta_1 t - \theta_2 t^2 + \theta_3 \log t\}, \quad 0 < t < \infty, \tag{1.1}$$

where C is a normalizing constant, and the (natural) parameter space is the union of $\{(\theta_1,\theta_2,\theta_3) : -\infty < \theta_1 < \infty, \theta_2 > 0, \theta_3 > -1\}$ and $\{(\theta_1,\theta_2,\theta_3) : \theta_1 > 0, \theta_2 = 0, \theta_3 > -1\}$. The special case $\theta_2 = 0$ gives the class of all gamma densities. The special case $\theta_2 > 0$, $\theta_3 = 0$ gives the class of all truncated normal densities with left truncation at 0. Usefulness of the family derives from its failure rate properties. (Cf. Glaser (1980).) Let $r(t) = f(t)/\bar{F}(t)$ denote the failure rate function, where $\bar{F}(t) = \int_t^\infty f(y)dy$. If $\theta_2 > 0$ and $\theta_3 \geq 0$, then $r(t)$ is monotone increasing (IFR); however, if $\theta_2 > 0$ and $\theta_3 < 0$, then $r(t)$ is bathtub shaped (BT): precisely, there exists $t_0 > 0$ such that $r(t)$ is strictly decreasing for $t < t_0$ and strictly increasing for $t > t_0$. For the gamma situation $\theta_2 = 0$, $r(t)$ is monotone decreasing (DFR) if $\theta_3 < 0$, constant if $\theta_3 = 0$ (the exponential density case), and monotone increasing if $\theta_3 > 0$. The family thus constitutes a rich class of models, encompassing all gamma distributions and all normal distributions left truncated at zero, as well as a variety of distributions having bathtub shaped failure rate functions. The latter property is of practical concern, since a bathtub failure rate corresponds to the common phenomenon of an item's improving with age initially and deteriorating with age eventually. In this paper statistical inference for the family will be considered with an emphasis on determining whether the model sampled fits the bathtub category.

Maximum likelihood estimation of the parameters based on random sampling is reviewed in Section 2. In Section 3 various hypothesis tests are proposed which address the question of whether a bathtub classification is appropriate. Finally, in Section 4 maximum likelihood estimation based on type II censoring is considered. Included is an estimate of the location of the bathtub's "plug."

2. MAXIMUM LIKELIHOOD ESTIMATION BASED ON RANDOM SAMPLING

Maximum likelihood estimation of the parameters under random sampling is discussed by Glaser (1978). The maximization is complicated by the different roles played by θ_1 depending on whether it is positive, zero, or negative. A re-parametrization motivated by $|\theta_1|$'s being basically a scale parameter proves effective. The overall family of models comprises the following four classes:

Class 1. $f(t) = [\lambda/\Gamma(\theta,\rho)]\exp\{-\lambda t-\theta(\lambda t)^2\}(\lambda t)^{\rho-1}$, $\lambda > 0, \theta > 0, \rho > 0$, (2.1)

where $\Gamma(\theta,\rho)$ is defined by $\Gamma(\theta,\rho) = \int_0^\infty \exp\{-y - \theta y^2\}y^{\rho-1}dy$.

Class 2. $f(t) = [\delta/\Lambda(\theta,\rho)]\exp\{\delta t - \theta(\delta t)^2\}(\delta t)^{\rho-1}$, $\delta > 0, \theta > 0, \rho > 0$, (2.2)

where $\Lambda(\theta,\rho)$ is defined by $\Lambda(\theta,\rho) = \int_0^\infty \exp\{y - \theta y^2\}y^{\rho-1}dy$.

Class 3. $f(t) = [2\lambda/\Gamma(\rho/2)]\exp\{-(\lambda t)^2\}(\lambda t)^{\rho-1}$, $\lambda > 0, \rho > 0$, (2.3)

where $\Gamma(\cdot)$ is the gamma function, defined by

$\Gamma(\xi) = \int_0^\infty \exp\{-y\}y^{\xi-1}dy$.

Class 4. $f(t) = [\lambda/\Gamma(\rho)]\exp\{-\lambda t\}(\lambda t)^{\rho-1}$, $\lambda > 0, \rho > 0$. (2.4)

Class 1 is formed from $\theta_1 > 0$, $\theta_2 > 0$; Class 2 is formed from $\theta_1 < 0$, $\theta_2 > 0$; Class 3 is formed from $\theta_1 = 0$, $\theta_2 > 0$; and Class 4 is formed from $\theta_1 > 0$, $\theta_2 = 0$. For Classes 1, 2, and 3, f is BT if $\rho < 1$, IFR if $\rho \geq 1$, and truncated normal if $\rho = 1$. Class 4 is the class of gamma densities, with f DFR if $\rho < 1$, IFR if $\rho > 1$, and exponential if $\rho = 1$.

As described by Glaser (1978), the maximum likelihood estimator of $(\theta_1,\theta_2,\theta_3)$ can be computed by the following algorithm. Let $\underset{\sim}{t} = (t_1,\ldots,t_n)$ denote the observed random sample. For each Class i, i=1,2,3,4, let $L_i(\underset{\sim}{t})$ denote the likelihood function based on the corresponding parametrization (for example, $L_1(\underset{\sim}{t}) = L_1(\underset{\sim}{t};\lambda,\theta,\rho) = \Pi_1^n f(t_j)$ with $f(t)$ taken from (2.1)), and let $\hat{L}_i(\underset{\sim}{t})$ denote the maximized likelihood for the class. Hence, for Class 1, if $(\hat{\lambda}^{(1)},\hat{\theta}^{(1)},\hat{\rho}^{(1)})$ denotes the MLE of (λ,θ,ρ) based on (2.1), then $\hat{L}_1(\underset{\sim}{t})$ is the maximized likelihood for Class 1, i.e., $\hat{L}_1(\underset{\sim}{t}) = L_1(\underset{\sim}{t};\hat{\lambda}^{(1)},\hat{\theta}^{(1)},\hat{\rho}^{(1)})$. The overall maximum likelihood is $\hat{L}(\underset{\sim}{t}) = \max_{1 \le i \le 4}\hat{L}_i(\underset{\sim}{t})$, and the MLE of $(\theta_1,\theta_2,\theta_3)$ is the value of $(\theta_1,\theta_2,\theta_3)$ corresponding to the MLE of the parameters in the class with the largest maximized likelihood. For example, if $\hat{L}(\underset{\sim}{t}) = \hat{L}_2(\underset{\sim}{t})$, then the MLE of $(\theta_1,\theta_2,\theta_3)$ is $(-\hat{\delta}^{(2)},\hat{\theta}^{(2)}(\hat{\delta}^{(2)})^2,\hat{\rho}^{(2)}-1)$.

Computational algorithms for the MLEs for the individual classes are presented by Glaser (1978). To review, let (s_1,s_2,s_3) denote $(\frac{1}{n}\Sigma\, t_j,\ \frac{1}{n}\Sigma\, t_j^2,\ \frac{1}{n}\Sigma \log t_j)$. Then $(\hat{\lambda}^{(1)},\hat{\theta}^{(1)},\hat{\rho}^{(1)})$ is the solution to the system of equations $s_1^2/s_2 = \Gamma^2(\theta,\rho+1)/\Gamma(\theta,\rho)\Gamma(\theta,\rho+2)$, $s_3 - \log s_1 = \psi(\theta,\rho) - \log[\Gamma(\theta,\rho+1)/\Gamma(\theta,\rho)]$, and $\lambda = \Gamma(\theta,\rho+1)/s_1\Gamma(\theta,\rho)$, where $\psi(\theta,\rho)$ is defined by $\psi(\theta,\rho) = (\partial/\partial\rho)\log\Gamma(\theta,\rho)$ $= [\Gamma(\theta,\rho)]^{-1}\int_0^\infty \exp\{-y-\theta y^2\}y^{\rho-1}\log y\,dy$. Similarly, $(\hat{\delta}^{(2)},\hat{\theta}^{(2)},\hat{\rho}^{(2)})$ is the solution to the system of equations $s_1^2/s_2 = \Lambda^2(\theta,\rho+1)/\Lambda(\theta,\rho)\Lambda(\theta,\rho+2)$, $s_3 - \log s_1 = \zeta(\theta,\rho) - \log[\Lambda(\theta,\rho+1)/\Lambda(\theta,\rho)]$, and $\delta = \Lambda(\theta,\rho+1)/s_1\Lambda(\theta,\rho)$, where $\zeta(\theta,\rho)$ is defined by $\zeta(\theta,\rho) = (\partial/\partial\rho)\log\Lambda(\theta,\rho)$ $= [\Lambda(\theta,\rho)]^{-1}\int_0^\infty \exp[y-\theta y^2]y^{\rho-1}\log y\,dy$. Also, $(\hat{\lambda}^{(3)},\hat{\rho}^{(3)})$ is the solution to the system of equations $s_3 - \frac{1}{2}\log s_2 = \frac{1}{2}\psi(\frac{\rho}{2}) - \frac{1}{2}\log(\frac{\rho}{2})$ and $\lambda^2 = \rho/2s_2$, where $\psi(\cdot)$ is the psi function, defined by $\psi(\xi) = (\partial/\partial\xi)\log\Gamma(\xi) = [\Gamma(\xi)]^{-1}\int_0^\infty \exp\{-y\}y^{\rho-1}\log y\,dy$. Finally, for the well-known gamma parametrization, $(\hat{\lambda}^{(4)},\hat{\rho}^{(4)})$ is the solution to the system of equations $s_3 - \log s_1 = \psi(\rho) - \log\rho$ and $\lambda = \rho/s_1$.

3. BATHTUB RELATED HYPOTHESIS TESTS

In this section assume $T_1, \ldots, T_n$ constitutes a random sample taken from some distribution in $\mathcal{F}$. Certain hypothesis tests regarding the condition of a bathtub failure rate function will be considered.

(a) Testing $H: \theta_2 = 0$ versus $K: \theta_2 > 0$.

The subclass of $\mathcal{F}$ restricted to $\theta_2 = 0$ represents the class of gamma distributions, each of which has a monotone failure rate function. The alternatives $\theta_2 > 0$ include bathtub failure rate functions ($\theta_3 < 0$). The form of optimum critical regions depends on whether θ_3 is known.

If θ_3 is known, a uniformly most powerful unbiased (UMPU) test exists based on the statistic $V = \Sigma T_j^2/(\Sigma T_j)^2$. Since for the gamma situation (i.e., $\theta_2 = 0$), V is distributed independently of ΣT_j (cf. Glaser (1976)), it follows from the general theory given by Lehmann (1959, Chapter 5) that the level α UMPU critical function is

$$\varphi(v) = \begin{cases} 1 & \text{if } v < C_0 \\ 0 & \text{if } v \geq C_0 \end{cases},$$

where C_0 is determined by the equation

$$\alpha = P_{\theta_2=0}[V < C_0]. \tag{3.1}$$

An (n-1)-fold integral representation of (3.1) involving the Dirichlet distribution allows computational evaluation of C_0. Assume $H: \theta_2 = 0$

holds. For convenience, denote $n-1$ by m and $\theta_3 + 1$ by ρ. Define $Z_i = T_i/\sum_1^n T_j$, $i=1,\ldots,m$. Then the variables $Z_1,\ldots,Z_m$ have a Dirichlet distribution with joint density

$$h(z_1,\ldots,z_m) = [\Gamma(n\rho)/\Gamma^n(\rho)]\Pi_1^m z_i^{\rho-1}(1-\sum_1^m z_i)^{\rho-1},$$

where $0 < z_i < 1$, $i=1,\ldots,m$, and $0 < \sum_1^m z_i < 1$. Since $V = \sum_1^m Z_i^2 + (1-\sum_1^m Z_i)^2$, (3.1) becomes

$$\alpha = \int_{D(C_0)} \cdots \int h(z_1,\ldots,z_m)dz_1\ldots dz_m, \tag{3.2}$$

where $D(C_0) = \{(z_1,\ldots,z_m) : 0 < z_i < 1,\ i=1,\ldots,m,\ 0 < \sum_1^m z_i < 1$, and $\sum_1^m z_i^2 + (1-\sum_1^m z_i)^2 < C_0\}$. Evaluation of C_0 by means of (3.2) is complicated by the awkward limits of integration. A tractable form of (3.2) is achieved upon applying an orthogonal transformation which incorporates $\sum_1^m z_i$ as a variable. Consider the $m \times m$ orthogonal matrix, $\underset{\sim}{0}$, obtained by the Gram-Schmidt orthogonalization process (cf. Kaplan (1952)), and defined by

$$\underset{\sim}{0} = \begin{bmatrix} m^{-\frac{1}{2}} & m^{-\frac{1}{2}} & m^{-\frac{1}{2}} & m^{-\frac{1}{2}} & \cdots & m^{-\frac{1}{2}} \\ -[m(m-1)]^{-\frac{1}{2}} & [(m-1)/m]^{\frac{1}{2}} & -[m(m-1)]^{-\frac{1}{2}} & -[m(m-1)]^{-\frac{1}{2}} & \cdots & -[m(m-1)]^{-\frac{1}{2}} \\ -[(m-1)(m-2)]^{-\frac{1}{2}} & 0 & [(m-2)/(m-1)]^{\frac{1}{2}} & -[(m-1)(m-2)]^{-\frac{1}{2}} & \cdots & -[(m-1)(m-2)]^{-\frac{1}{2}} \\ -[(m-2)(m-3)]^{-\frac{1}{2}} & 0 & 0 & [(m-3)/(m-2)]^{\frac{1}{2}} & \cdots & -[(m-2)(m-3)]^{-\frac{1}{2}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2^{-\frac{1}{2}} & 0 & 0 & 0 & \cdots & 2^{-\frac{1}{2}} \end{bmatrix}$$

The transformation $\underset{\sim}{u} = \underset{\sim}{0}\,\underset{\sim}{z}$ gives

$$u_1 = m^{-\frac{1}{2}} z_1 + \ldots + m^{-\frac{1}{2}} z_m$$

$$u_2 = -[m(m-1)]^{-\frac{1}{2}} z_1 + [(m-1)/m]^{\frac{1}{2}} z_2 - [m(m-1)]^{-\frac{1}{2}} z_3 - \ldots - [m(m-1)]^{-\frac{1}{2}} z_m$$

$$u_3 = -[(m-1)(m-2)]^{-\frac{1}{2}} z_1 + [(m-2)/(m-1)]^{\frac{1}{2}} z_3 - [(m-1)(m-2)]^{-\frac{1}{2}} z_4 - \ldots - [(m-1)(m-2)]^{-\frac{1}{2}} z_m$$

$$\vdots$$

$$u_k = -[(m-k+2)(m-k+1)]^{-\frac{1}{2}} z_1 + [(m-k+1)/(m-k+2)]^{\frac{1}{2}} z_k - [(m-k+2)(m-k+1)]^{-\frac{1}{2}} z_{k+1} - \ldots - [(m-k+2)(m-k+1)]^{-\frac{1}{2}} z_m$$

$$\vdots$$

$$u_m = -2^{-\frac{1}{2}} z_1 + 2^{-\frac{1}{2}} z_m.$$

The restrictions, $0 < z_i < 1$, $i=1,\ldots,m$, and $0 < \sum_1^m z_i < 1$, translate to $u_i \in A_i$, $i=1,\ldots,m$, where $A_1 = (0, m^{-\frac{1}{2}})$, $A_2 = (-[m(m-1)]^{-\frac{1}{2}}, [(m-1)/m]^{\frac{1}{2}})$, $A_3 = (-[(m-1)(m-2)]^{-\frac{1}{2}}, [(m-2)/(m-1)]^{\frac{1}{2}}), \ldots, A_m = (-2^{-\frac{1}{2}}, 2^{\frac{1}{2}})$. The condition $\sum_1^m z_i^2 + (1 - \sum_1^m z_i)^2 < C_0$ imposes marginally on u_1 the restriction $m^{\frac{1}{2}}/n - (1/n)(nC_0 - 1)^{\frac{1}{2}} < u_1 < m^{\frac{1}{2}}/n + (1/n)(nC_0 - 1)^{\frac{1}{2}}$. Consequently, (3.2) assumes the computable form

$$\alpha = \int_{(u_1,\ldots,u_m)\,:\,u_i \in A_i \cap B_i(C_0)} \cdots \int [\Gamma(n\rho)/\Gamma^n(\rho)][\Pi_1^m z_i(u_1,\ldots,u_m)]^{\rho-1}(1-m^{\frac{1}{2}}u_1)^{\rho-1} du_m \ldots du_1, \quad (3.3)$$

where $B_1(C_0) = (m^{\frac{1}{2}}/n - (1/n)(nC_0 - 1)^{\frac{1}{2}}, m^{\frac{1}{2}}/n + (1/n)(nC_0 - 1)^{\frac{1}{2}})$,

$B_i(C_0) = (0, C_0 - [\Sigma_{j=1}^{i-1} u_j^{\,2} + (1 - m^{\frac{1}{2}} u_1)^2])$, $i=2,\ldots,m$, and $z_i(u_1,\ldots,u_m)$ is obtained from $\underset{\sim}{z} = \underset{\sim}{O}'\underset{\sim}{u}$. For fixed C_0, the right-hand-side of (3.3) can be computed by numerical integration. Since the right-hand-side increases with C_0, the exact value of C_0 corresponding to a given level α can be found by an easy trial and error process.

A large sample approximation of C_0 is readily obtained from multivariate central limit theorems. Assume $H : \theta_2 = 0$ holds so that $T_1,\ldots,T_n$ constitutes a random sample from a gamma distribution; for convenience, denote $\theta_3 + 1$ by ρ and θ_1 by λ. Since

$$E_{\theta_2=0}(T^k) = (\rho + k-1)\ldots\rho/\lambda^k \qquad \text{for } k=1,2,3,4,$$

it follows from the multivariate Lindeberg-Levy central limit theorem that under $H : \theta_2 = 0$, $n^{\frac{1}{2}}(\underset{\sim}{Y}_n - \underset{\sim}{\mu}) \xrightarrow{\mathcal{L}} N_2(\underset{\sim}{0}, \underset{\sim}{\Sigma})$, where

$$\underset{\sim}{Y}_n = \begin{pmatrix} Y_{1n} \\ Y_{2n} \end{pmatrix} = \begin{pmatrix} \frac{1}{n}\Sigma T_j^{\,2} \\ \frac{1}{n}\Sigma T_j \end{pmatrix},\ \underset{\sim}{\mu} = \begin{pmatrix} \rho(\rho+1)\lambda^2 \\ \rho/\lambda \end{pmatrix},\ \text{and}\ \underset{\sim}{\Sigma} = \begin{pmatrix} 2\rho(\rho+1)(2\rho+3)/\lambda^4 & 2\rho(\rho+1)/\lambda^3 \\ 2\rho(\rho+1)/\lambda^3 & \rho/\lambda^2 \end{pmatrix}.$$

By applying the transformation $g(\underset{\sim}{Y}_n) = Y_{1n}/Y_{2n}^2$, it follows (cf. Rao (1973, p. 387)) that

$$n^{\frac{1}{2}}(g(\underset{\sim}{Y}_n) - g(\underset{\sim}{\mu})) \xrightarrow{\mathcal{L}} N(0, \underset{\sim}{B}\underset{\sim}{\Sigma}\underset{\sim}{B}'), \quad \text{where}$$

$\underset{\sim}{B} = (\partial g(\underset{\sim}{Y}_n)/\partial Y_{1n}, \partial g(\underset{\sim}{Y}_n)/\partial Y_{2n})\big|_{(\underset{\sim}{Y}_n = \underset{\sim}{\mu})} = (\lambda^2/\rho^2, -2\lambda(\rho+1)/\rho^2)$. That is, $n^{\frac{1}{2}}(nV_n - (\rho+1)/\rho) \xrightarrow{\mathcal{L}} N(0, 2(\rho+1)/\rho^3)$, where $V = V_n$. A large sample approximation of C_0 is therefore

$$C_0 \doteq \frac{1}{n}\left\{\frac{\rho+1}{\rho} - z_\alpha\left[\frac{1}{n}\cdot 2(\rho+1)/\rho^3\right]^{1/2}\right\},$$

where z_α is defined by $\Phi(z_\alpha) = 1-\alpha$. In computing C_0 by the exact multiple integral representation (3.3), a logical starting point would be this large sample approximation.

If θ_3 is unknown, the UMPU test is conditional on the statistics $(T,S) = (\Sigma T_j, \Sigma \log T_j)$. Denote ΣT_j^2 by U. From the general theory of Lehmann (1959 , Chapter 4), the critical function of the level α UMPU test is

$$\varphi(u,t,s) = \begin{cases} 1 & \text{if } u < C_0(t,s) \\ 0 & \text{if } u \geq C_0(t,s) \end{cases},$$

where $C_0(t,s)$ is determined by the equation

$$\alpha = P_{\theta_2=0}\{U < C_0(t,s) \mid (T,S) = (t,s)\}. \tag{3.4}$$

Consider the conditional distribution of $(T_1,\ldots,T_n)$, given $(T,S) = (t,s)$, assuming $H: \theta_2 = 0$ holds. Again for convenience denote θ_1 by λ and $\theta_3 + 1$ by ρ. The joint density of $(T_1,\ldots,T_n)$ is $[\lambda^{n\rho}/\Gamma^n(\rho)]e^{(\rho-1)s-\lambda t}$.

Because of the independence of $\frac{1}{n}T$ and $(e^{S/n}/\frac{1}{n}T)^n$, it follows from Glaser (1976) that the joint density of (T,S) is

$$[\lambda^{n\rho}/\Gamma^n(\rho)][(2\pi)^{(n-1)/2}n^{1/2}/\Gamma(\tfrac{n-1}{2})]t^{-1}e^{\rho s-\lambda t}(n\log\tfrac{t}{n}-s)^{(n-3)/2}\xi_n(e^s/(\tfrac{t}{n})^n),$$

for a certain function $\xi_n(\cdot)$. Consequently, the conditional density of $(T_1,\ldots,T_n)$, given $(T,S) = (t,s)$, is

$$[\Gamma(\tfrac{n-1}{2})/(2\pi)^{(n-1)/2}n^{1/2}]te^{-s}(n\log\tfrac{t}{n}-s)^{-(n-3)/2}\xi_n^{-1}(e^s/(\tfrac{t}{n})^n).$$

The essential point is that the conditional density is constant (i.e., uniform) in $(t_1,\ldots,t_n)$ over the region for which $\Sigma t_j = t$ and $\Sigma \log t_j = s$. The right-hand-side of (3.4) is thus the ratio of (n-2)-dimensional volumes,

$$\frac{\text{Volume}\{(t_1,\ldots,t_n) : \Sigma t_j^2 < C_0(t,s), \Sigma t_j = t, \Sigma \log t_j = s\}}{\text{Volume}\{(t_1,\ldots,t_n) : \Sigma t_j = t, \Sigma \log t_j = s\}}.$$

Unfortunately, computation of these volumes appears prohibitively complicated. An exception is the rather uninteresting case n=3, where a cubic equation can be obtained which generates C_0.

A large sample approximation for $C_0(t,s)$ can be found by methods similar to those used in the known θ_3 case. Let

$\underset{\sim}{Y}_n = (\frac{1}{n}\Sigma T_j^2, \frac{1}{n}\Sigma T_j, \frac{1}{n}\Sigma \log T_j)'$. It follows from the multivariate Lindeberg-Levy central limit theorem that under $H : \theta_2 = 0$,

$$n^{\frac{1}{2}}(\underset{\sim}{Y}_n - \underset{\sim}{\mu}) \xrightarrow{\mathcal{L}} N_3(\underset{\sim}{0}, \underset{\sim}{\Sigma}),$$

$$\text{where } \underset{\sim}{\mu} = \begin{pmatrix} \rho(\rho+1)/\lambda^2 \\ \rho/\lambda \\ \psi(\rho)-\log\lambda \end{pmatrix} \quad \text{and } \underset{\sim}{\Sigma} = \begin{pmatrix} 2\rho(\rho+1)(2\rho+3)/\lambda^4 & 2\rho(\rho+1)/\lambda^3 & (2\rho+1)/\lambda^2 \\ 2\rho(\rho+1)/\lambda^3 & \rho/\lambda^2 & 1/\lambda \\ (2\rho+1)/\lambda^2 & 1/\lambda & \psi'(\rho) \end{pmatrix}$$

More importantly, the relevant conditional distribution converges to normality. Partition $\underset{\sim}{Y}_n$, $\underset{\sim}{\mu}$, and $\underset{\sim}{\Sigma}$ as

$$\underset{\sim}{Y}_n = \begin{pmatrix} \frac{1}{n}U_n \\ \hline \frac{1}{n}T_n \\ \frac{1}{n}S_n \end{pmatrix}, \quad \underset{\sim}{\mu} = \begin{pmatrix} \mu^{(1)} \\ \hline \underset{\sim}{\mu}^{(2)} \end{pmatrix} = \begin{pmatrix} \rho(\rho+1)/\lambda^2 \\ \hline \rho/\lambda \\ \psi(\rho)-\log\lambda \end{pmatrix}, \text{ and}$$

$$\underset{\sim}{\Sigma} = \left(\begin{array}{c|c} \Sigma_{11} & \underset{\sim}{\Sigma}_{12} \\ \hline \underset{\sim}{\Sigma}_{21} & \underset{\sim}{\Sigma}_{22} \end{array}\right) = \left(\begin{array}{c|cc} 2\rho(\rho+1)(2\rho+3)/\lambda^4 & 2\rho(\rho+1)\lambda^3 & (2\rho+1)/\lambda^2 \\ \hline 2\rho(\rho+1)/\lambda^3 & \rho/\lambda^2 & 1/\lambda \\ (2\rho+1)/\lambda^2 & 1/\lambda & \psi'(\rho) \end{array}\right)$$

From the work of Steck (1959), it follows that the limiting conditional distribution of $n^{\frac{1}{2}}(\frac{1}{n}U_n - \mu^{(1)})$, given $n^{\frac{1}{2}}\left(\begin{pmatrix} \frac{1}{n}T_n \\ \frac{1}{n}S_n \end{pmatrix} - \underset{\sim}{\mu}^{(2)}\right) = n^{\frac{1}{2}}\left(\begin{pmatrix} \frac{1}{n}t_n \\ \frac{1}{n}s_n \end{pmatrix} - \underset{\sim}{\mu}^{(2)}\right)$, i.e., given $\begin{pmatrix} T_n \\ S_n \end{pmatrix} = \begin{pmatrix} t_n \\ s_n \end{pmatrix}$, is univariate normal

with mean $\mu = n^{\frac{1}{2}} \underset{\sim}{\Sigma}_{12} \underset{\sim}{\Sigma}_{22}^{-1} \left(\begin{pmatrix} \frac{1}{n}t_n \\ \frac{1}{n}s_n \end{pmatrix} - \underset{\sim}{\mu}^{(2)}\right)$ and variance $\sigma^2 = \Sigma_{11} - \underset{\sim}{\Sigma}_{12} \underset{\sim}{\Sigma}_{22}^{-1} \underset{\sim}{\Sigma}_{21}$.

The difficulty introduced by the fact that the asymptotic conditional distribution depends on the unknown parameters ρ and λ will be overcome by substituting estimators which are consistent under H. As shown by Neyman (1955), such substitution does not affect the asymptotic properties of the test. When $H: \theta_2 = 0$ is true, the (consistent) MLE, $(\hat{\rho}, \hat{\lambda})$, of (ρ, λ) is the simultaneous solution to the equation $\begin{pmatrix} \frac{1}{n} t_n \\ \frac{1}{n} s_n \end{pmatrix} = \underset{\sim}{\mu}^{(2)}$ (cf. Class 4 of Section 2). Consequently, the (consistent) MLE, $\hat{\mu}$, of μ under H is $\hat{\mu} = 0$. On the other hand, computation of σ^2 yields $\sigma^2 = (\rho/\lambda^4)[2(\rho+1) - (\rho\psi'(\rho)-1)^{-1}]$, so that the (consistent) MLE, $\widehat{\sigma^2}$, of σ^2 under H is $\widehat{\sigma^2} = (\hat{\rho}/\hat{\lambda}^4)[2(\hat{\rho}+1) - (\hat{\rho}\psi'(\hat{\rho}) - 1)^{-1}]$. Finally, the (consistent) MLE, $\widehat{\mu^{(1)}}$, of $\mu^{(1)}$ under H is $\widehat{\mu^{(1)}} = \hat{\rho}(\hat{\rho}+1)/\hat{\lambda}^2$. To summarize, if $H: \theta_2 = 0$ is true, the approximate conditional distribution of $n^{\frac{1}{2}}(\frac{1}{n}U_n - \widehat{\mu^{(1)}})/\hat{\sigma}$, given $\begin{pmatrix} T_n \\ S_n \end{pmatrix} = \begin{pmatrix} t_n \\ s_n \end{pmatrix}$, is normal with mean 0 and variance 1. A large sample approximation of the level α critical value $C_0(t,s)$, from (3.4), is therefore $C_0(t,s) = n\widehat{\mu^{(1)}} - n^{\frac{1}{2}} z_\alpha \hat{\sigma}$.

(b) Testing $H: \theta_3 \geq 0$ versus $K: \theta_3 < 0$.

If $\theta_2 > 0$, the subclass of $\mathfrak{F}$ restricted to $\theta_3 \geq 0$ has only IFR functions, whereas the subclass categorized by $\theta_3 < 0$ has exclusively BT functions. The test of H versus K, assuming $\theta_2 > 0$, is therefore a test of increasing failure rate against bathtub alternatives. For the case $\theta_2 = 0$, i.e., the gamma distribution, the test of H versus K is equivalent to testing IFR versus DFR. A UMPU test for this situation is provided by Glaser (1973). For the remainder of this section it will be assumed that θ_2 is positive but unknown.

Based on the notation $(U,T,S) = (\Sigma T_j^2, \Sigma T_j, \Sigma \log T_j)$, the level α UMPU test is conditional on (T,U):

$$\varphi(s,t,u) = \begin{cases} 1 & \text{if } s < C_0(t,u) \\ 0 & \text{if } s \geq C_0(t,u) \end{cases},$$

where $C_0(t,u)$ is determined by the equation

$$\alpha = P_{\theta_3=0}\{S < C_0(t,u) \mid (T,U) = (t,u)\}. \tag{3.5}$$

When $\theta_3 = 0$, the sample $T_1,\dots,T_n$ is truncated normal; however, the conditional distribution of S given $(T,U) = (t,u)$ does not appear tractable. Fortunately, a large sample approximation of $C_0(t,u)$ can be found by employing the method used in testing $\theta_2 = 0$ with θ_3 unknown. Let

$$\underset{\sim}{Y}_n = \begin{pmatrix} \frac{1}{n}\Sigma \log T_j \\ \frac{1}{n}\Sigma T_j \\ \frac{1}{n}\Sigma T_j^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{n} S_n \\ \frac{1}{n} T_n \\ \frac{1}{n} U_n \end{pmatrix}.$$

From the multivariate Lindeberg-Levy central limit theorem it follows that if $\theta_3 = 0$, then $n^{\frac{1}{2}}(\underset{\sim}{Y}_n - \underset{\sim}{\mu}) \xrightarrow{\mathcal{L}} N_3(\underset{\sim}{0}, \underset{\sim}{\Sigma})$, where $\underset{\sim}{\mu}$ is the mean vector and $\underset{\sim}{\Sigma}$ the covariance matrix of $(\log T_j, T_j, T_j^2)'$, assuming $\theta_3 = 0$. Partition $\underset{\sim}{Y}_n$ as

$$\underset{\sim}{Y}_n = \begin{pmatrix} \frac{1}{n} S_n \\ ---- \\ \frac{1}{n} T_n \\ \frac{1}{n} U_n \end{pmatrix}$$, and partition $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ accordingly. Then (cf. Steck (1959)), for $\theta_3 = 0$ the relevant asymptotic conditional distribution is normal: i.e., the conditional distribution of $n^{\frac{1}{2}}(\frac{1}{n} S_n - \widehat{\mu^{(1)}})/\hat{\sigma}$, given $\binom{T_n}{U_n} = \binom{t_n}{u_n}$, tends to N(0,1), where $\widehat{\mu^{(1)}}$ and $\widehat{\sigma^2}$ denote the respective (consistent) MLEs of $\mu^{(1)}$ and $\sigma^2 = \Sigma_{11} - \underset{\sim}{\Sigma}_{12} \underset{\sim}{\Sigma}_{22}^{-1} \underset{\sim}{\Sigma}_{21}$, computed under the assumption that $\theta_3 = 0$. The large sample approximation of $C_0(t,u)$, from (3.5), is therefore $C_0(t,n) = n \widehat{\mu^{(1)}} - n^{\frac{1}{2}} z_\alpha \hat{\sigma}$.

Computation of $\widehat{\mu^{(1)}}$ and $\hat{\sigma}$ is somewhat tedious. The MLE, $(\widehat{\theta_1}, \widehat{\theta_2})$, of (θ_1, θ_2) is the simultaneous solution to the equation

$$\begin{pmatrix} \frac{1}{n} t_n \\ \frac{1}{n} u_n \end{pmatrix} = \underset{\sim}{\mu}^{(2)} = \begin{pmatrix} E_{\theta_3=0} T_j \\ E_{\theta_3=0} T_j^2 \end{pmatrix} = \begin{pmatrix} (C - \theta_1)/2\theta_2 \\ [1 - \theta_1(C - \theta_1)/2\theta_2]/2\theta_2 \end{pmatrix},$$

where, from (1.1), $C = C(\theta_1, \theta_2, 0) = \left\{(\pi/\theta_2)^{\frac{1}{2}} \exp(\theta_1^2/4\theta_2)[1 - \Phi(\theta_1(2\theta_2)^{-\frac{1}{2}})]\right\}^{-1}$.

Suppose for definiteness that $\binom{t_n}{u_n}$ leads to a positive value for $\widehat{\theta_1}$. By the invariance property of maximum likelihood, the MLEs, $\widehat{\mu^{(1)}}$ and $\hat{\sigma}$, are obtainable from the parametrization (inspired by Class 1 of Section 2), $[\lambda/\Gamma(\theta,1)]\exp\{-\lambda t - \theta(\lambda t)^2\}$, $\lambda > 0$, $\theta > 0$. Here $\lambda = \theta_1$ and $\theta = \theta_2/\theta_1^2$, so that $\hat{\lambda} = \theta_1$ and $\hat{\theta} = \hat{\theta}_2/\hat{\theta}_1^2$. Since this parametrization gives $\mu^{(1)} = \int_0^\infty [\lambda/\Gamma(\theta,1)]\exp\{-\lambda t - \theta(\lambda t)^2\}\log t\, dt = \psi(\theta,1) - \log\lambda$, and

$$
\Sigma = \left(\begin{array}{c|cc}
\psi'(\theta,1) & \frac{1}{\lambda}\frac{\Gamma(\theta,2)}{\Gamma(\theta,1)}[\psi(\theta,2)-\psi(\theta,1)] & \frac{1}{\lambda^2}\frac{\Gamma(\theta,3)}{\Gamma(\theta,1)}[\psi(\theta,3)-\psi(\theta,1)] \\
\hline
\frac{1}{\lambda}\frac{\Gamma(\theta,2)}{\Gamma(\theta,1)}[\psi(\theta,2)-\psi(\theta,1)] & \frac{1}{\lambda^2}\frac{\Gamma(\theta,1)\Gamma(\theta,3)-\Gamma^2(\theta,2)}{\Gamma^2(\theta,1)} & \frac{1}{\lambda^3}\frac{\Gamma(\theta,1)\Gamma(\theta,4)-\Gamma(\theta,2)\Gamma(\theta,3)}{\Gamma^2(\theta,1)} \\
\frac{1}{\lambda^2}\frac{\Gamma(\theta,3)}{\Gamma(\theta,1)}[\psi(\theta,3)-\psi(\theta,1)] & \frac{1}{\lambda^3}\frac{\Gamma(\theta,1)\Gamma(\theta,4)-\Gamma(\theta,2)\Gamma(\theta,3)}{\Gamma^2(\theta,1)} & \frac{1}{\lambda^4}\frac{\Gamma(\theta,1)\Gamma(\theta,5)-\Gamma^2(\theta,3)}{\Gamma^2(\theta,1)}
\end{array}\right),
$$

it follows that $\widehat{\mu^{(1)}} = \psi(\hat{\theta},1) - \log\hat{\lambda}$ and, after a bit of computation, $\hat{\sigma} = \{\psi'(\hat{\theta},1) - [g_2{}^2(p_2-p_1)^2(g_1g_5-g_3{}^2) - 2g_2g_3(p_2-p_1)(p_3-p_1)(g_1g_4-g_2g_3) + g_3{}^2(p_3-p_1)^2(g_1g_3-g_2{}^2)]/[g_1g_3-g_2{}^2)(g_1g_5-g_3{}^2) - (g_1g_4-g_2g_3)^2]\}^{1/2}$, where $g_i \equiv \Gamma(\hat{\theta},i)$ and $p_i \equiv \psi(\hat{\theta},i)$. Analogous computations of $\widehat{\mu^{(1)}}$ and $\hat{\sigma}$ are possible based on $\hat{\theta}_1$ negative or $\hat{\theta}_1$ zero. For negative $\hat{\theta}_1$ a parametrization of the Class 2 variety would be used; for $\hat{\theta}_1 = 0$, a Class 3 parametrization would be used.

4. CENSORED SAMPLING

(a) Maximum likelihood estimation of the parameters based on type II censored sampling.

A common sampling technique in life testing is type II censoring at r out of n: n items are simultaneously put on test; the lifetimes of only the first r items to fail are recorded. In effect, only the first r order statistics, $T_{(1)} < \ldots < T_{(r)}$, are observed from the random sample $T_1,\ldots,T_n$, of n lifetimes. To estimate the parameters, $(\theta_1,\theta_2,\theta_3)$, of the sampled distribution in $\mathfrak{F}$ by the method of maximum likelihood, the basic strategy described in Section 2 for random sampling can be used;

i.e., likelihood maximization is carried out individually for each of the four classes, whereupon the largest of the four maxima provides the MLE $(\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3)$. For the sake of brevity, only Class 1 will be considered. Maximization for the other classes is analogous, and in the case of Class 4 (the gamma distribution), already worked out by Wilk et al. (1962).

For convenience, denote by Y_j the j^{th} order statistic $T_{(j)}$. The (Class 1) likelihood function is

$$L = \frac{n!}{(n-r)!}\,\frac{\lambda^r}{\Gamma^r(\theta,\rho)}\exp\left\{-\sum_1^r \lambda y_j - \theta\sum_1^r (\lambda y_j)^2\right\}\left[\prod_1^r(\lambda y_j)\right]^{\rho-1}\left[\int_{y_r}^{\infty}\frac{\lambda}{\Gamma(\theta,\rho)}\exp\left\{-\lambda t-\theta(\lambda t)^2\right\}(\lambda t)^{\rho-1}dt\right]^{n-r}.$$

Following the general approach of Wilk et al. (1962), define $\xi = \lambda y_r$, $p = (\prod_1^r y_j)^{1/r}/y_r$, $s = \sum_1^r y_j/ry_r$, and $t = \sum_1^r y_j^{\,2}/ry_r^{\,2}$. Then

$$\log L = \log\left[\frac{n!}{(n-r)!}\,\frac{1}{y_r^{\,r}}\right]$$
$$+\, n\rho\log\xi - n\log\Gamma(\theta,\rho) + (r\rho-r)\log p - r\xi s - r\theta\xi^2 t$$
$$+\,(n-r)\log J(\theta,\rho,\xi),$$

where $J(\theta,\rho,\xi) = \int_1^{\infty} x^{\rho-1}\exp\{-\xi x - \theta(\xi x)^2\}dx$. The equations $(\partial/\partial\rho)\log L = 0$, $(\partial/\partial\theta)\log L = 0$, and $(\partial/\partial\xi)\log L = 0$ are equivalent, respectively, to the equations

$$p = \exp\left\{\frac{1}{r}\left[\frac{n}{\Gamma(\theta,\rho)}\frac{\partial}{\partial\rho}\Gamma(\theta,\rho) - n\log\xi - (n-r)\left(\frac{\partial}{\partial\rho}J(\theta,\rho,\xi)\right)/J(\theta,\rho,\xi)\right]\right\} \tag{4.1}$$

$$t = (r\xi^2)^{-1}\left[\frac{n}{\Gamma(\theta,\rho)}\frac{\partial}{\partial\theta}\Gamma(\theta,\rho) + (n-r)\xi^2 J(\theta,\rho+2,\xi)/J(\theta,\rho,\xi)\right] \tag{4.2}$$

$$s = r^{-1}\left[2r\theta\xi t - n\rho/\xi + (n-r)\left[J(\theta,\rho+1,\xi) + 2\theta\xi J(\theta,\rho+2,\xi)\right]/J(\theta,\rho,\xi)\right], \tag{4.3}$$

where $(\partial/\partial\rho)\Gamma(\theta,\rho) = \int_0^\infty \exp\{-y - \theta y^2\}y^{\rho-1}\log y\,dy$ and $(\partial/\partial\rho)J(\theta,\rho,\xi)$ $= \int_1^\infty x^{\rho-1}\exp\{-\xi x - \theta(\xi x)^2\}\log x\,dx$. The MLEs of ρ, θ, and ξ are defined by the simultaneous solution of the equations (4.1), (4.2) and (4.3). (Note that the MLE of λ is the MLE of ξ divided by y_r.) However, an inverse iteration scheme for determining the MLEs is more convenient: for any assigned ρ, θ, and ξ, use (4.1) to calculate p, (4.2) to calculate t, and (4.3) with t to calculate s. Continue with the iteration process until values of (ρ, θ, ξ) are assigned which give resultant (p,t,s) calculations equal (to a desired degree of accuracy) to the actual observed values.

(b) Maximum likelihood estimation of the plug.

For a model in $\mathfrak{F}$ having a bathtub shaped failure rate function a parameter of interest is the point, τ, known as the plug, where the failure rate function takes on its minimum value, i.e., the point of transition which separates decreasing failure rate from increasing failure rate. Let $g(\cdot)$ denote the reciprocal of the failure rate function. Then, as shown by Glaser (1980),

$$g'(t) = (\theta_1 + 2\theta_2 t - \theta_3/t)\left[\frac{\int_t^\infty \exp\{-\theta_1 x - \theta_2 x^2 + \theta_3 \log x\}dx}{\exp\{-\theta_1 t - \theta_2 t^2 + \theta_3 \log t\}}\right] - 1. \qquad (4.4)$$

If $\theta_2 > 0$ and $\theta_3 < 0$, the model has a bathtub shaped failure rate function, and the plug τ is the unique value of t satisfying $g'(t) = 0$. Since in this case, $g'(t) > 0$ for $t < \tau$ and $g'(t) < 0$ for $t > \tau$, computation of τ from (4.4) may proceed by a swift trial and error scheme. Similarly, if the MLEs, $\hat{\theta}_2$ and $\hat{\theta}_3$, obtained either by random sampling or censored sampling, satisfy $\hat{\theta}_2 > 0$ and $\hat{\theta}_3 < 0$, then the MLE, $\hat{\tau}$, of τ is the solution to $\widehat{g'}(\tau) = 0$, where (4.4) is used with $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)$ substituted for $(\theta_1, \theta_2, \theta_3)$.

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